Viscous Flow with Second-Order Slip Velocity over a Stretching Sheet

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In this paper, viscous flow with a second-order slip condition over a permeable stretching surface is solved analytically. The current work differs from the previous studies in the application of a new second-order slip velocity model. The closed form solution reported is an exact solution of the full governing Navier-Stokes equations. The effects of slip and mass transfer parameters are discussed.

Key words: Similarity Solution; Stretching Surface; Navier-Stokes Equations; Analytical Solution; Exact Solution; Slip Flow.

1. Introduction

The flow induced by a moving boundary is important in the extrusion processes [1 – 3]. Sakiadis [4, 5] studied the boundary layer flow on a continuously stretching surface moving with a constant speed. However, according to Wang [6], Sakiadis' solution was not an exact solution of the Navier-Stokes (NS) equations. Crane [7] presented an exact solution of the two-dimensional NS equations for a stretching sheet problem. The effect of mass transfer on the Crane flow was investigated by Gupta and Gupta [8]. The stretching boundary problem was extended by Wang [9] to a three-dimensional situation. The result of Wang for a stretching disk was extended to rotating flow over a stretchable disk by Fang [10] and for flow between two stretchable disks by Fang and Zhang [11]. It is interesting to note that all these solutions are exact solutions of the NS equations. In the recent years, micro-scale fluid dynamics in the micro-electro-mechanical systems (MEMS) have received considerable research attention. Because of the micro-scale dimensions, the fluid flow behaviour belongs to the slip flow regime and differs significantly from the traditional no-slip flow [12]. However, the flow in the slip regime still obeys the Navier-Stokes equations. In addition, partial velocity slips over a moving surface also occur for fluids with particles such as emulsions, suspensions, foams, and polymer solutions [13]. The slip flows for different flow configurations have been studied in the literature [14 – 20]. Among these papers, the slip flow over an impermeable stretching surface was studied by Andersson [14] and Wang [15]. In a recent paper, the mass transfer effect in slip flow over a stretching surface was reported by Wang [19]. However, in these papers, only first-order Maxwell slip conditions were used. Recently, Wu [21] proposed a new second-order slip velocity model which matches better with the Fukui-Kaneko results based on the direct numerical simulation of the linearized Boltzmann equation [22]. Most recently, this new slip model was applied to the flow over a shrinking sheet and significantly different flow behaviour was found compared with the first-order slip velocity [23]. The objective of the current paper is to study the slip flow over a permeable stretching surface with the newly proposed Wu's slip velocity model. Exact solutions of the governing NS equations are presented and discussed.

2. Mathematical Formulation

Consider a steady, two-dimensional laminar flow over a continuously stretching sheet in an incompressible quiescent fluid. The sheet stretching velocity is $u_w = U_0 x$ with U_0 being a positive constant and the wall mass transfer velocity is $v_w = v_w(x)$, which will be determined later. The x-axis runs along the stretching surface and the y-axis is perpendicular to it. The governing NS equations of this problem read [14 – 15]

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{1}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right), \quad (2)$$

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right), \quad (3)$$

with the boundary conditions (BCs)

$$u(x,0)=U_0x+U_{\text{slip}}, \quad v(x,0)=v_w(x), \quad \text{and} \quad u(x,\infty)=0, \quad (4)$$

where u and v are the velocity components in the x and y-directions, respectively, ν is the kinematic viscosity, p is the fluid pressure, ρ is the fluid density, and u_{slip} is the slip velocity at the surface and it is negative due to stretching. Wu's slip velocity model used in this paper is valid for arbitrary Knudsen numbers and is given as follows [21]:

$$U_{\text{slip}}=\frac{2}{3}\left(\frac{3-\alpha l^3}{\alpha}-\frac{3}{2}\frac{1-l^2}{K_n}\right)\lambda\frac{\partial u}{\partial y} -\frac{1}{4}\left[l^4+\frac{2}{K_n^2}(1-l^2)\right]\lambda^2\frac{\partial^2 u}{\partial y^2}=A\frac{\partial u}{\partial y}+B\frac{\partial^2 u}{\partial y^2}, \quad (5)$$

where $l=\min\left[\frac{1}{K_n},1\right]$, α is the momentum accommodation coefficient with $0\le\alpha\le 1$, λ is the molecular mean free path, and K_n is the Knudsen number defined as the mean free path λ divided by a characteristic length for the flow. Based on the definition of l, it is seen that for any given value of K_n, we have $0\le l\le 1$. The molecular mean free path is always positive. Thus we know that $B<0$ and A is a positive number. In order to obtain a similarity equation, the stream function and the similarity variable can be posited in the following form:

$$\psi(x,y)=f(\eta)x\sqrt{\nu U_0}, \quad \text{and} \quad \eta=y\sqrt{\frac{U_0}{\nu}}. \quad (6)$$

With these definitions, the velocities are expressed as $u=U_0xf'(\eta)$ and $v=-\sqrt{U_0\nu}f(\eta)$. The wall mass transfer velocity becomes $v_w(x)=-\sqrt{U_0\nu}f(0)$. The similarity equation is obtained as

$$f'''+ff''-f'^2=0 \quad (7)$$

with the BCs as

$$f(0)=s, \quad f'(0)=1+\gamma f''(0)+\delta f'''(0), \quad \text{and} \quad f'(\infty)=0, \quad (8)$$

where s is the wall mass transfer parameter showing the strength of the mass transfer at the surface, γ is the first-order velocity slip parameter with $0<\gamma=A\sqrt{\frac{U_0}{\nu}}$, and δ is the second-order velocity slip parameter with $0>\delta=\frac{BU_0}{\nu}$. The pressure term can be obtained from (3) as $\frac{p}{\rho}=\nu\frac{\partial v}{\partial y}-\frac{v^2}{2}$ + const.

3. Results and Discussion

In this paper, we derive a closed form exact solution of (7) subject to the BCs of (8). We assume a solution of the form $f(\eta)=a+be^{-\beta\eta}$. The application of the BCs (8) give the solution for a and b as

$$b=-\frac{1}{\beta+\gamma\beta^2-\delta\beta^3}, \quad (9)$$

$$a=s+\frac{1}{\beta+\gamma\beta^2-\delta\beta^3}. \quad (10)$$

Substituting the assumed solution into (7) yields $a=\beta$. The use of this relationship in (10) leads to the following fourth-order algebraic equation for β:

$$\delta\beta^4-(\gamma+\delta s)\beta^3+(\gamma s-1)\beta^2+s\beta+1=0. \quad (11)$$

Then the solution reads

$$f(\eta)=s+\frac{1}{\beta+\gamma\beta^2-\delta\beta^3}-\frac{1}{\beta+\gamma\beta^2-\delta\beta^3}e^{-\beta\eta} \quad (12)$$

and

$$f'(\eta)=\frac{1}{1+\gamma\beta-\delta\beta^2}e^{-\beta\eta}. \quad (13)$$

Based on the results in (13), it is easy to show that $f''(0)=-\frac{\beta}{1+\gamma\beta-\delta\beta^2}$. Now we will focus on the solution of (11). For $\delta=0$ with a first-order slip velocity, the solution reduces to the solution given by Wang [19]. For both $\delta=0$ and $s=0$, the solution reduces to those by Andersson [14] and Wang [15]. Thus (12) is a more general solution. Since the solution depends on β, we focus on the solution of (11). The solution of β is determined by the wall mass transfer parameter, the first-order slip parameter γ, and the second-order slip parameter δ, and only positive roots of (11) give physically meaningful solutions. Equation (11) is a complete quartic equation and can be written in the following form:

$$\beta^4-\frac{(\gamma+\delta s)}{\delta}\beta^3+\frac{(\gamma s-1)}{\delta}\beta^2+\frac{s}{\delta}\beta+\frac{1}{\delta} =\beta^4+a_3\beta^3+a_2\beta^2+a_1\beta+a_0=0. \quad (14)$$

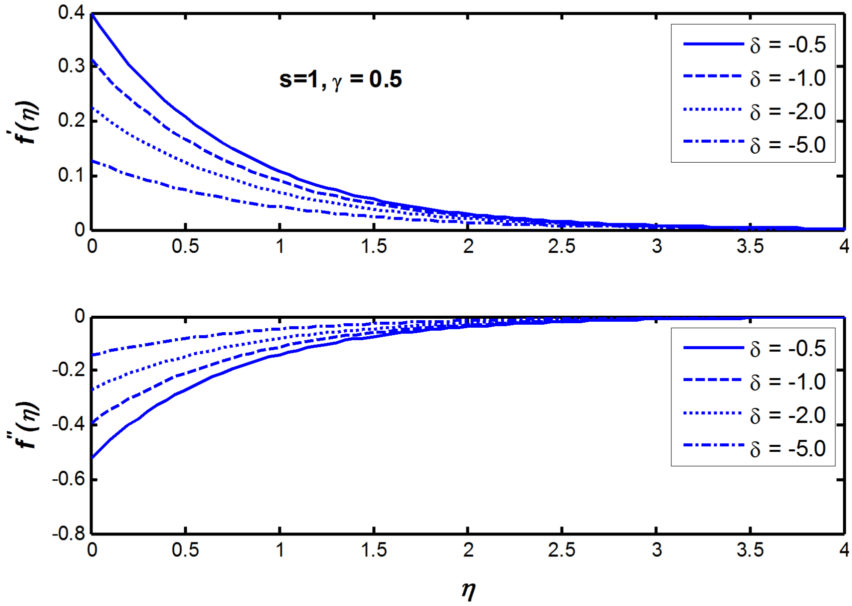

Fig. 1. Dimensionless velocity and shear stress profiles at $s=1$ and $\gamma=0.5$ for different second-order slip parameters.

Equation (14) can be converted to the following standard form by defining $z=\beta+\frac{a_3}{4}$:

$$z^4+pz^2+qz+r=0, \tag{15}$$

where $p=a_2-\frac{3}{8}a_3^2$, $q=a_1-\frac{1}{2}a_2a_3+\frac{1}{8}a_3^3$, and $r=a_0-\frac{1}{4}a_1a_3+\frac{1}{16}a_2a_3^2-\frac{3}{256}a_3^4$. There are four roots of (15); however, for negative values of δ, only one positive real root exists. Therefore there is only one physically feasible solution of (11). The physical solution can be given in the following explicit form:

$$\beta=\frac{\sqrt{C}}{2}+\frac{1}{2}\sqrt{D-\frac{2q}{\sqrt{C}}}-\frac{a_3}{4}$$
$$\text{or}\quad \beta=-\frac{\sqrt{C}}{2}+\frac{1}{2}\sqrt{D+\frac{2q}{\sqrt{C}}}-\frac{a_3}{4}, \tag{16}$$

where

$$C=-\frac{2p}{3}+[2^{\frac{1}{3}}(p^2+12r)]\left[3\left(2p^2+27q^2-72pr\right.\right.$$
$$\left.\left.+\sqrt{-4(p^2+12r)^3+(2p^2+27q^2-72pr)^2}\right)^{\frac{1}{3}}\right]^{-1}$$
$$+\left[\left(2p^2+27q^2-72pr\right.\right.$$
$$\left.\left.+\sqrt{-4(p^2+12r)^3+(2p^2+27q^2-72pr)^2}\right)^{\frac{1}{3}}\right]$$
$$\cdot[2^{\frac{1}{3}}\cdot 3]^{-1}$$

Table 1. Solution of β for $\delta=-0.5$ and different values of s and γ.

s	γ 0	0.5	1	3	10
−3	0.2914	0.2629	0.2425	0.1951	0.1351
−2	0.3891	0.3458	0.3161	0.2497	0.1699
−1	0.5564	0.4910	0.4469	0.3505	0.2375
0	0.8556	0.7709	0.7130	0.5847	0.4304
1	1.3743	1.3057	1.2599	1.1657	1.0754
2	2.1418	2.1106	2.0907	2.0526	2.0213
3	3.0576	3.0458	3.0380	3.0226	3.0093

Table 2. Solution of β for $\delta=-2.0$ and different values of s and γ.

s	γ 0	0.5	1	3	10
−3	0.2677	0.2472	0.2313	0.1906	0.1342
−2	0.3446	0.3173	0.2960	0.2420	0.1683
−1	0.4709	0.4359	0.4080	0.3355	0.2345
0	0.7071	0.6695	0.6384	0.5530	0.4230
1	1.2103	1.1899	1.1731	1.1280	1.0672
2	2.0517	2.0470	2.0430	2.0320	2.0169
3	3.0173	3.0160	3.0149	3.0118	3.0068

Table 3. Solution of β for $\delta=0.0$ and different values of s and γ.

s	γ 0	0.5	1	3	10
−3	0.3028	0.2695	0.2470	0.1967	0.1355
−2	0.4142	0.3593	0.3247	0.2526	0.1704
−1	0.6180	0.5214	0.4656	0.3563	0.2385
0	1.0000	0.8393	0.7549	0.5982	0.4331
1	1.6180	1.4142	1.3247	1.1852	1.0787
2	2.4142	2.2143	2.1479	2.0672	2.0233
3	3.3028	3.1249	3.0796	3.0327	3.0107

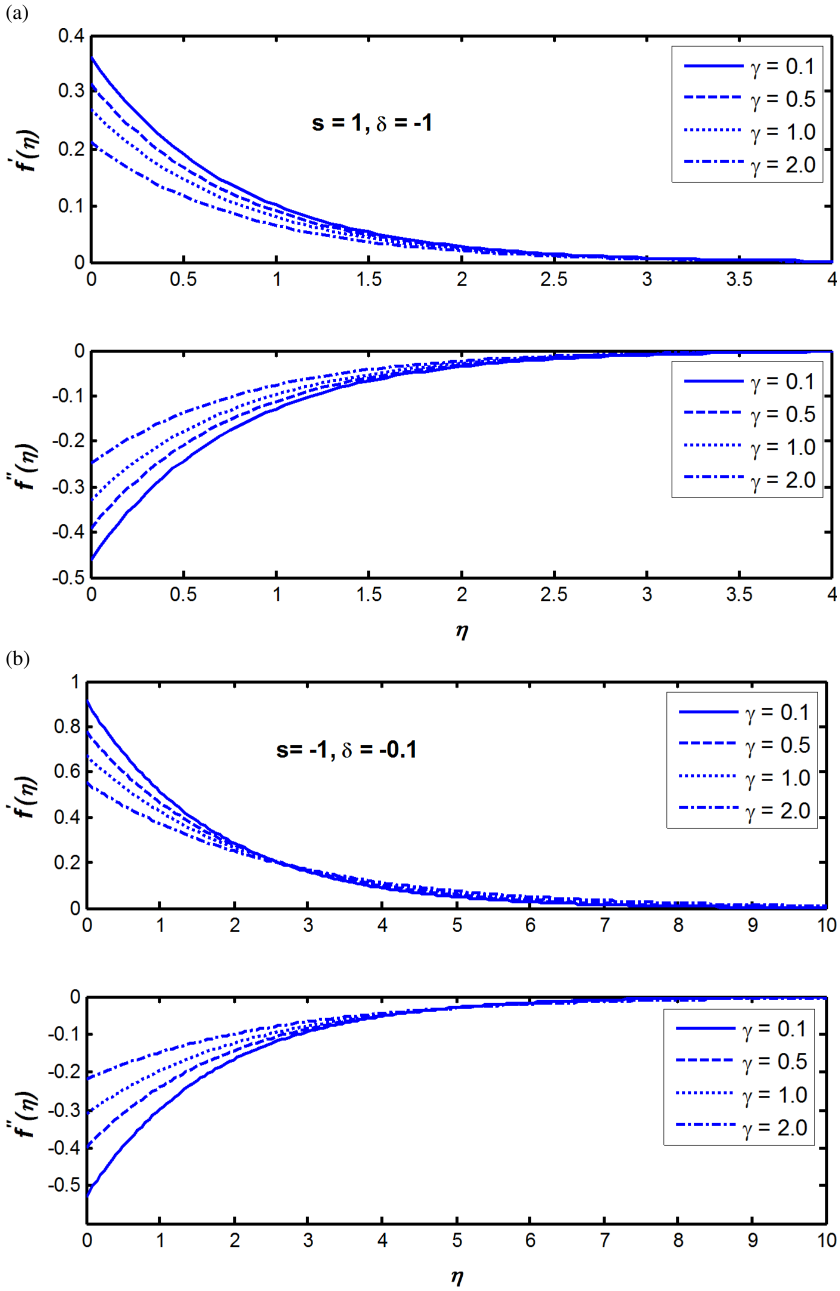

Fig. 2. Dimensionless velocity and shear stress profiles at $\delta = -1$ and $s = 1$ (a) and $\delta = -0.1$ and $s = -1$ (b) for different first-order slip parameters.

and

$$D = -\frac{4p}{3} - [2^{\frac{1}{3}}(p^2 + 12r)]\Bigg[3\Bigg(2p^2 + 27q^2 - 72pr - \sqrt{-4(p^2+12r)^3 + (2p^2+27q^2-72pr)^2}\Bigg)^{\frac{1}{3}}\Bigg]^{-1} - \Bigg[\Bigg(2p^2 + 27q^2 - 72pr + \sqrt{-4(p^2+12r)^3 + (2p^2+27q^2-72pr)^2}\Bigg)^{\frac{1}{3}}\Bigg] \cdot [2^{\frac{1}{3}} \cdot 3]^{-1}.$$

There is only one root from the two forms valid. However, the choice of the right solution depends on the parameter domain. For $q = 0$ or $p = q = 0$, (16) might not hold. The solution can be further simplified. For

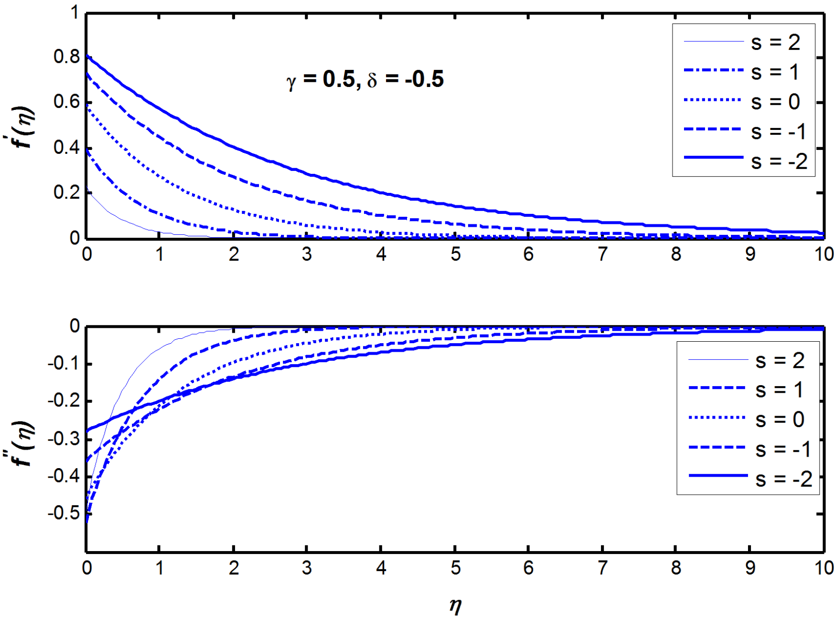

Fig. 3. Dimensionless velocity and shear stress profiles at $\delta = -0.5$ and $\gamma = 0.5$ for different mass suction parameters.

$p = q = 0$, we find that $r = a_0 - \frac{1}{256}a_3^4$. Since $\delta < 0$, it is seen that $r < 0$. Then the solution will be $\beta = (-r)^{\frac{1}{4}}$. For $q = 0$, the solution will be

$$\beta = \sqrt{-\frac{p}{2} - \frac{\sqrt{p^2-4r}}{2}} - \frac{a_3}{4}$$
$$\text{or} \quad \beta = \sqrt{-\frac{p}{2} + \frac{\sqrt{p^2-4r}}{2}} - \frac{a_3}{4}. \tag{17}$$

In order to show the dependence of β on the three parameters, some results will be presented and discussed in this section. Some numerical values of β for different combination values of γ and s are shown in Tables 1 and 2 for $\delta = -0.5$ and $\delta = -2.0$, respectively. For comparison, the corresponding results for $\delta = 0$ are shown in Table 3. It is seen that the value of β increases with the increase of mass suction parameter and decreases with the increase of the first-order slip parameter. However, it decreases with the decrease of the second-order slip parameter. From (14) for δ with a large magnitude, we find that $\beta = s$ for positive mass suction parameter.

In order to show the effects of the three parameters on the flow field and the shear stresses, some examples will be illustrated. Some typical velocity and shear stress profiles are shown in Figures 1, 2 and 3 for different combinations of the three parameters. Figure 1 shows the effects of the second-order slip parameter by keeping the first-order slip and mass suction parameters constant. It is seen that the wall slip velocity increases with the increase of $|\delta|$ and the wall drag force decreases with the increase of $|\delta|$. There is no crossover point among different velocity and shear stress profiles for different values of δ. The effects of first-order velocity slip parameters are shown in Figure 2. In Figure 2a, a mass suction is applied at the surface with $\delta = -1$ and $s = 1$. It is found that higher velocity slips occur for a larger value of γ. The wall drag force becomes smaller for a large value γ. Interesting velocity and shear stress profiles are seen in Figure 2b with $s = -1$ and $\delta = -0.1$. Again, the velocity slip increases and the wall drag force decreases with the increase of the first-order slip parameter. However, it is interesting that there are crossover points for both the velocity and the shear stress profiles. Under mass injection, the velocity profiles damp faster for a smaller value of the first-order slip parameter and the flow penetration into the fluid actually becomes shorter. Mass injection helps the flow to penetrate deeper into the fluid. In Figure 3, the effects of mass transfer parameters are illustrated for $\delta = -0.5$ and $\gamma = 0.5$. It is seen that a higher value of the mass suction parameter increases the wall slip velocity and the wall shear stress. From the above examples of the velocity and shear stress profiles, it is seen that the combination of the two slip parameters and the mass transfer can greatly change the fluid flow and the shear stresses on the surface and in the fluid. The introduction of the second-order slip parameter can increase the wall slip velocity and reduce the wall drag, which is a better match to the real flow physics compared with the previous velocity slip models [21].

4. Conclusion

In summary, the viscous fluid flow under new velocity slip conditions over a continuously stretching permeable surface is solved in a closed analytical form. The solution is an exact solution of the governing NS equations. There is only one physical solution for any combination of the two slip parameters and the mass transfer parameter. The combined effects of the two slips and mass transfer parameters greatly influence the fluid flow and shear stresses on the wall and in the fluid.

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